stichting
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AFDELING INFORMATICA
(DEPARTMENT OF COMPUTER SCIENCE)

IW 237/83 SEPTEMBER

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ON SPECIFYING SETS OF INTEGERS

Preprint

kruislaan 413 1098 SJ amsterdam

Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification 68B15, 68C01
1982 CR. Categories: D.2.1, D.3.1, E.0

On specifying sets of integers[*)]

by

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ABSTRACT

We consider the problem of deriving an algebraic specification for a rather simple set-theoretical data type called $SOI_{\#}$. $SOI_{\#}$ is merely a collection of finite sets of integers equipped with an operator to insert a number into a set and another to determine the cardinality of a set. We show $SOI_{\#}$ has a finite conditional specification, but no finite equational specification, under the initial algebra semantics for specifications invented by the ADJ Group.

KEY WORDS & PHRASES: *set-theoretical data types, initial algebra semantics, equational and conditional specifications*

[*)] To appear in E.I.K.

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0. INTRODUCTION

Set-theoretical data structures play an important role as an example subject in data type specification theory. Axiomatisation of data types in general can be done in several ways. For instance, as in [3], one can use sets of equations or sets of conditional equations under initial algebra semantics. But also a method called *structural recursion*, introduced in KLAEREN [4], can be applied for specifying data types.

In [5] Klaeren considers the example of a collection of finite sets of non-negative integers equipped with an operator to insert a natural number into a setand another to determine the cardinality of a set (Example 3.2). This special set-theoretical data type is modelled as a two-sorted algebra

$$SOI_{\#} = ((\omega;S,\underline{0}),(SETS;\underline{\emptyset}),IN,\underline{\#}),$$

an algebra with natural numbers and sets of these as sorts and successor S on the set $\underline{\omega}$ of natural numbers, insertion IN of a natural number into a set, and cardinality $\underline{\#}$ of a set as operators.

Klaeren uses this example to show that it can be treated by his method of structual recursion, although as he presumes, it has no finite equational specification under initial algebra semantics.

In this paper we shall prove that his presumption is right, but also that a finite specification *can* be made, if *conditional* equations are allowed:

<u>THEOREM</u>. $SOI_{\#}$ *has a finite conditional specification but fails to posses a finite equational specification.*

We shall prove the first statement in Section 2, the second in Section 3. Section 1 contains some preliminary material.

The theorem just stated is also another neat example indicating the difference in power of equations and conditional equations as means of specifications of infinite data structures. (In [1] the use of conditional equations in *final* algebraic specification is exploited, but without proof that conditionals are essentially needed. That conditional equations also have more power that equations in specifying *finite* data structures, is shown in [2], where a certain result on specification is obtained much easier when one is allowed to use also conditionals.)

In the following we shall assume that the reader is familiar with the work of the ADJ Group, at least up to the level of their basic paper [3]. Knowledge of Klaeren's work is obviously desirable but not formally necessary.

We thank J.V. Tucker and J.W. Klop for discussions and suggestions concerning this paper.

1. PRELIMINARIES

Initial algebra semantics assigns to a specification (Σ,E) in which Σ is a signature, i.e. a set of names of operators and individual constants, and E is a set of equations, a unique meaning in the class $ALG(\Sigma,E)$ of all Σ-algebras satisfying the equations of E in the following way: two terms t and t' over Σ are identical iff t and t' can be *proved* equal from the axioms in E. (See [1,3,7]; the semantics of conditional equations is given in [6].)

We shall now resume the main notions of initial algebra specification.

An (n-*sorted*) *algebra* V of signature Σ is a structure $(V_1,\ldots,V_n;\Sigma)$ in which the V_i are sets of elements, called the *domains* of V and Σ is a set of symbols naming functions σ which are each defined on some cartesian product of the V_i:

$$\sigma\colon V_{\ell_1} \times \ldots \times V_{\ell_k} \to V_m \text{ where } 1 \le \ell_1,\ldots,\ell_k,\ m \le n,$$

and naming special elements of the V_i, the so-called *individual constants* of V. Σ is called the *signature* of V naming the *constants* of V.

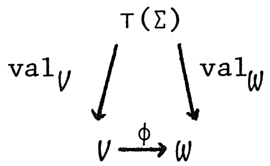
The following facts hold:
Let V and W be algebras of signature Σ, both finitely generated by their constants (i.e. V and W are *minimal*).
Then: (1) any Σ-homomorphism $\phi\colon V \to W$ is surjective.
(2) if ϕ, $\psi\colon V \to W$ are Σ-homomorphisms then $\phi = \psi$.
(3) if there are Σ-homomorphisms $\phi\colon V \to W$ and $\psi\colon W \to V$ then $V \cong W$ (by either ϕ or ψ).

Let $\equiv$ be an equivalence relation on the n-sorted algebra V; then we call a family of sets $J_i \subseteq V_i$ $(1 \le i \le n)$ such that $\forall b \in V_i\ \exists_1\ a \in J_i$ for which

$b \equiv a$, a *traversal* for $\equiv$.

If Σ is a signature, then $T(\Sigma)$ denotes the Σ-algebra of all terms over Σ and $T_\Sigma\ [X_1,\dots,X_p]$ denotes the algebra of polynomials in the indeterminates $X_1,\dots,X_p$. For convenience we let $T(\Sigma) \subseteq T_\Sigma[\vec{X}]$ for every vector $\vec{X}$.

If $\mathcal{V}$ is a Σ-algebra, then we mean by *term evaluation in* $\mathcal{V}$ a map $val_\mathcal{V}$: $T(\Sigma) \to \mathcal{V}$ which evaluates each term $t \in T(\Sigma)$ by substituting the constants of $\mathcal{V}$ for their names in t. The map $val_\mathcal{V}$ can be uniquely defined as an epimorphism $T(\Sigma) \to \mathcal{V}$. If ϕ: $\mathcal{V} \to \mathcal{W}$ is a homomorphism between Σ-algebras, then the following diagram commutes:



We define *polynomial evaluation* in $\mathcal{V}$ as the substitution of some $\vec{a} = (a_1,\dots,a_p) \in (\cup_{i=1}^n V_i)^p$ for indeterminates $\vec{X} = (X_1,\dots,X_p)$ (where a_i is an element of that domain over which X_i ranges), and of the constants of $\mathcal{V}$ for their names into polynomial $t(\vec{X}) \in T_\Sigma[X_1,\dots,X_p]$ followed by the evaluation of $t(\vec{a})$ in $\mathcal{V}$.

An *equation* is a pair $(t(\vec{X}),t'(\vec{X}))$ of polynomials from some $T_\Sigma[X_1,\dots,X_p]$ written as $t(\vec{X}) = t'(\vec{X})$, where it must be noted that $t(\vec{X})$ and $t'(\vec{X})$ need not have any indeterminate in common.

A *conditional equation* is a formula of the form

$$\bigwedge_i (t_i(\vec{X}) = t_i'(\vec{X})) \to t(\vec{X}) = t'(\vec{X}).$$

If E is a set of (conditional) equations over Σ and $\mathcal{V}$ is a Σ-algebra such that $\mathcal{V} \models E$, we say that $\mathcal{V}$ is an E-*algebra*. We define ALG(Σ,E) as the class of all E-algebras and $T(\Sigma,E)$ as the *initial algebra* for ALG(Σ,E), constructed from $T(\Sigma)$; $T(\Sigma,E) = T(\Sigma)/\equiv_E$ where $\equiv_E$ denotes the smallest congruence on $T(\Sigma)$ that identifies terms of $T(\Sigma)$ by means of the equations of E. If $t \in T(\Sigma)$ we mean by $C_E(t)$ the $\equiv_E$-equivalence class $\in T(\Sigma,E)$ that contains t.

An algebra V of signature Σ_V has a finite *equational (conditional) specification* (Σ,E) if $\Sigma_V = \Sigma$, E is a finite set of (conditional) equations over Σ, and $T(\Sigma,E) \cong V$. More details can be found in [1,3,6,7].

2. A CONDITIONAL SPECIFICATION OF $SOI_\#$

We shall begin this section with repeating the formal definition of the structure $SOI_\#$:

$$SOI_\# = ((\omega;S,\underline{0}),(SETS;\underline{\emptyset}),IN,\underline{\#})$$

in which ω is the set of natural numbers,
SETS is the class of finite sets of natural numbers,
S names the successor function $s\colon \omega \to \omega$ defined by $s(x) = x+1$,
IN names the insertion function in: $\omega \times SETS \to SETS$ defined by
in $(x,\xi) = \xi \cup \{x\}$ where $x \in \omega$ and $\xi \in SETS$,
$\underline{\#}$ names the cardinality function $\#$: SETS $\to \omega$ defined by
$\#(\xi)$ = the number of elements in $\xi \in SETS$,
$\underline{0}$ names $0 \in \omega$, and
$\underline{\emptyset}$ names the empty set $\emptyset \in SETS$.

($SOI_\#$ can also be written in the usual form $(V_1,\ldots,V_n;\Sigma)$ as $(\omega,SETS;S,IN,\underline{\#},\underline{0},\underline{\emptyset})$, but the notation above is more suggestive of the working of the operators and the relation of the constants to the domains and is therefore preferable.)

Now we can prove the following.

THEOREM. $SOI_\#$ *has a finite conditional specification.*

PROOF. We shall prove the theorem by showing that a specific set E_0 of conditional equations specifies $SOI_\#$.

In the following we shall denote variables over ω by X, Y (with indices if necessary) and variables over SETS by Ξ (with indices if necessary), and refer to term $t \in T(\Sigma)$ with $val_{SOI_\#}(t) \in \omega$ *as first sort terms* and to terms $\tau \in T(\Sigma)$ with $val_{SOI_\#}(\tau) \in SETS$ as *second sort terms.*

In E_0 we take the equations:

(e_1) $IN(X,IN(X,\Xi)) = IN(X,\Xi)$

(e_2) $IN(X,IN(Y,\Xi)) = IN(Y,IN(X,\Xi))$

(e_3) $\alpha(X,Y) \wedge \beta(X,\Xi) \wedge \beta(Y,\Xi) \rightarrow \underline{\#}(IN(X,IN(Y,\Xi))) = SS(\underline{\#}(\Xi))$

where $\alpha(X,Y)$ stands for the formula

$\underline{\#}(IN(X,IN(Y,\underline{\emptyset}))) = SS(\underline{0})$

thus expressing in equational language the statement $X \neq Y$, and $\beta(X,\Xi)$ stands for the formula

$\underline{\#}(IN(X,\Xi)) = S(\underline{\#}(\Xi))$

expressing in equational language the statement $X \notin \Xi$.

(e_4) $\underline{\#}(IN(X,\underline{\emptyset})) = S(\underline{0})$

(e_5) $\underline{\#}(\underline{\emptyset}) = \underline{0}$

(e_6) $\underline{\#}(IN(\underline{0},IN(S(X),\underline{\emptyset}))) = SS(\underline{0})$

(e_7) $\underline{\#}(IN(S(X),IN(S(Y),\underline{\emptyset}))) = \underline{\#}(IN(X,IN(Y,\underline{\emptyset})))$.

Note that $SOI_{\#} \models E_0$ (proof by inspection of cases).

In order to prove that this E_0 specifies $SOI_{\#}$, i.e. $SOI_{\#} \cong T(\Sigma,E_0)$ where $\Sigma = \{S,IN,\underline{\#},\underline{0},\underline{\emptyset}\}$, we shall show first that the pair (J_1,J_2) of sets with

$$J_1 = \{S^i(\underline{0}) \mid i \in \omega\}$$

and

$$J_2 = \bigcup_{n=0}^{\infty} \{IN(S^{i_1^n}(\underline{0}),(\ldots,IN(S^{i_n^n}(\underline{0}),\underline{\emptyset})\ldots)) \mid i_1^n < \ldots < i_n^n \in \omega\}$$

(where n in i_k^n is an upper index and not an exponent!), is a traversal for $\equiv_{E_0}$ on $T(\Sigma)$.

For notational convenience we shall abbreviate

$$IN(S^{i_1}(\underline{0}),IN(S^{i_2}(\underline{0}),\ldots,IN(S^{i_n}(\underline{0}),\underline{\emptyset})\ldots))$$

as

$$IN(i_1,IN(i_2,\ldots,IN(i_n,\underline{\emptyset})\ldots)).$$

So we must prove that for every first sort term t there is a unique $a \in J_1$ such that $t \equiv_{E_0} a$ and for every second sort term τ there is a unique $\alpha \in J_2$ such that $\tau \equiv_{E_0} \alpha$, and to show this we must establish the following statements:

(i) $i \neq j \Rightarrow S^i(\underline{0}) \not\equiv_{E_0} S^j(\underline{0})$

(ii) Let $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_m$. Then:

$$\mathrm{IN}(i_1,\mathrm{IN}(i_2,\ldots,\mathrm{IN}(i_n,\underline{\emptyset})\ldots)) \equiv_{E_0} \mathrm{IN}(j_1,\mathrm{IN}(i_2,\ldots,\mathrm{IN}(j_m,\underline{\emptyset})\ldots))$$

$$\Leftrightarrow n = m \text{ and } i_k = j_k (1\leq k\leq n).$$

(iii) for every first sort term $t \in T(\Sigma)$, there is an $a \in J_1$ such that $a \equiv_{E_0} t$.

(iv) for every second sort term $\tau \in J(\Sigma)$, there is an $\alpha \in J_2$ such that $\alpha \equiv_{E_0} \tau$.

(i) and (ii) are trivial, because they are true in $\mathrm{SOI}_{\#}$, which was a model for E_0; (iii) and (iv) can easily be seen by applying the equations (e_1) up to (e_7) to rewrite an arbitrary term into one in the J_i.

Now we define the function ϕ: $\mathrm{SOI}_{\#} \rightarrow T(\Sigma,E_0)$ as follows:

$$\phi(z) = \begin{cases} C_{E_0}(S^z(\underline{0})) \text{ if } z \in \omega \\ C_{E_0}(\mathrm{IN}(r_1(z),\ldots,\mathrm{IN}(r_m(z),\underline{\emptyset})\ldots)) \text{ if } z \in \mathrm{SETS} \text{ and } r_1(z)<\ldots<r_m(z) \\ \text{are the elements of } z\text{, in increasing order and after deleting} \\ \text{equal elements.} \end{cases}$$

Obviously this function ϕ is bijective. It is also a homomorphism, for it holds that

(1) $S\phi(z) \equiv_{E_0} \phi(s(z))$ (for all $z \in \omega$) (trivial),

(2) $\mathrm{IN}(\phi(z),\phi(\xi)) \equiv_{E_0} \mathrm{IN}(S^z(\underline{0}),\mathrm{IN}(r_1(\xi),\ldots,r_n(\xi),\underline{\emptyset})) \equiv_{E_0}$

$\mathrm{IN}(r_1(\xi\cup\{z\}),\ldots,\mathrm{IN}(r_m(\xi\cup\{z\}),\underline{\emptyset})\ldots) = \phi(\xi\cup\{z\}) = \phi(in(z,\xi))$, for all $z \in \omega$ and $\xi \in \mathrm{SETS}$

(3) $\underline{\#}\phi(\xi) \equiv_{E_0} \phi(\#(\xi))$ (for all $\xi \in \mathrm{SETS}$), because for $\xi = \emptyset$:

$$\underline{\#}\phi(\underline{\emptyset}) \equiv_{E_0} \underline{\#}(\underline{\emptyset}) \equiv_{E_0} \underline{0} = \phi(0) = \phi(\#(\emptyset)), \text{ and for } \xi \neq \emptyset:$$

$$\underline{\#}(\mathrm{IN}(r_1(\xi),\ldots,\mathrm{IN}(r_m(\xi),\underline{\emptyset})\ldots)) \equiv_{E_0} S^m(\underline{0}) = \phi(m) = \phi(\#(\xi)).$$

Consequently $SOI_{\#} \cong T(\Sigma,E_0)$ (by ϕ), which we had to prove. □

3. $SOI_{\#}$ HAS NO EQUATIONAL SPECIFICATION

In this section we prove the following

<u>THEOREM</u>. $SOI_{\#}$ *can not be specified by means of some finite equational specification.*

<u>PROOF</u>. In the following we shall make use of the same terminology as we used in Section 2. Furthermore we shall call polynomials $t(\vec{X},\vec{\Xi}) \in T_{\Sigma}[\vec{X},\vec{\Xi}]$ with $\vec{X} = (X_1,\ldots,X_p)$ and $\vec{\Xi} = (\Xi_1,\ldots,\Xi_q)$, that produce first sort terms if first sort terms are substituted for the X_i and second sort terms for the Ξ_i, *first sort polynomials*. Analogously we define *second sort polynomials* $\tau(\vec{X},\vec{\Xi})$. E.g. $S^k(X)$ and $S^{\ell}\underline{\#}(\mathrm{IN}(X,\mathrm{IN}(X,\Xi)))$ are first sort polynomials, and $\mathrm{IN}(X,\underline{\emptyset})$ and $\mathrm{IN}(\underline{0},\mathrm{IN}(X,\Xi))$ are second sort polynomials.

In order to prove our theorem we start with a finite equational specification, say (Σ,E), where $\Sigma = \{S,\mathrm{IN},\underline{\#},\underline{0},\underline{\emptyset}\}$ and E is some finite set of equations that is sound in the sense that $SOI_{\#} \models E$. We will show that $T(\Sigma,E) \ncong SOI_{\#}$.

E defines the usual congruence relation $\equiv_E$ on $T(\Sigma) \times T(\Sigma)$. Next we define the congruence relation $\equiv_{SOI}$ on $T(\Sigma) \times T(\Sigma)$:

$$t \equiv_{SOI} t' \text{ iff } val_{SOI_{\#}}(t) = val_{SOI_{\#}}(t')$$

for each $t, t' \in T(\Sigma)$.
(In the sequel we shall abbreviate $val_{SOI_{\#}}(t)$ to val (t).) Further, for each $N \in \omega$ we define the congruence relation $\equiv_N$ on $T(\Sigma) \times T(\Sigma)$ by:

(1) $\equiv_N$ is reflexive, symmetrical and transitive.

(2) For all second sort terms τ_1 and τ_2: $\tau_1 \equiv_N \tau_2$ iff $\tau_1 \equiv_{SOI} \tau_2$.

(3) $S^{\ell}(\underline{\#}(\tau)) \equiv_N S^{k+\ell}(\underline{0})$ for every $\ell \in \omega$ iff $\underline{\#}(\tau) \equiv_{SOI} S^k(\underline{0})$ and $k \le N$ (i.e. $\#(val(\tau)) = k \le N$), where τ is a second sort term and $k \in \omega$.

(4) $S^k(\underline{\#}(\tau_1)) \equiv_N S^{\ell}(\underline{\#}(\tau_2))$ iff $S^k(\underline{\#}(\tau_1)) \equiv_{SOI} S^{\ell}(\underline{\#}(\tau_2))$ and $\#(val(\tau_1)) > N$, $\#(val(\tau_2)) > N$, where τ_1, τ_2 are second sort terms and $k,\ell \in \omega$.

This relation $\equiv_N$ has the following properties:

(i) $\bigcup_{N\in\omega} \equiv_N = \equiv_{SOI}$

(ii) $\equiv_N \subsetneq \equiv_{N+1}$.

Property (i) is obvious, in (ii) the inclusion is also obvious.
Claim 1, the inclusion in (ii) is strict.

Proof of claim 1:

For each $N \in \omega$ we define the structure $SOI^N_{\#} = ((\omega \cup \tilde{\omega}_N; S,\underline{0},\underline{\widetilde{N+1}}), (SETS;\underline{\emptyset}), IN,\underline{\#})$, where $\tilde{\omega}_N = \{\widetilde{N+1},\widetilde{N+2},\ldots\}$; S names the function s_N defined by

$$s_N(x) = x+1 \quad (x\in\omega)$$

and

$$s_N(\tilde{x}) = \widetilde{x+1} \quad (x\in\omega, x>N);$$

$\underline{0}$ names $0 \in \omega$; $\widetilde{N+1}$ names $\widetilde{N+1} \in \tilde{\omega}_N$; SETS is again the class of finite sets of natural numbers $\in \omega$; $\underline{\emptyset}$ names the empty set $\emptyset \in$ SETS, $\underline{\#}$ names the function $\#_N$: SETS $\to (\omega \cup \tilde{\omega}_N)$ defined by

$$\#_N(\xi) = \begin{cases} \#(\xi) \text{ if } \#(\xi) \le N \\ \widetilde{\#(\xi)} \text{ if } \#(\xi) > N \end{cases}$$

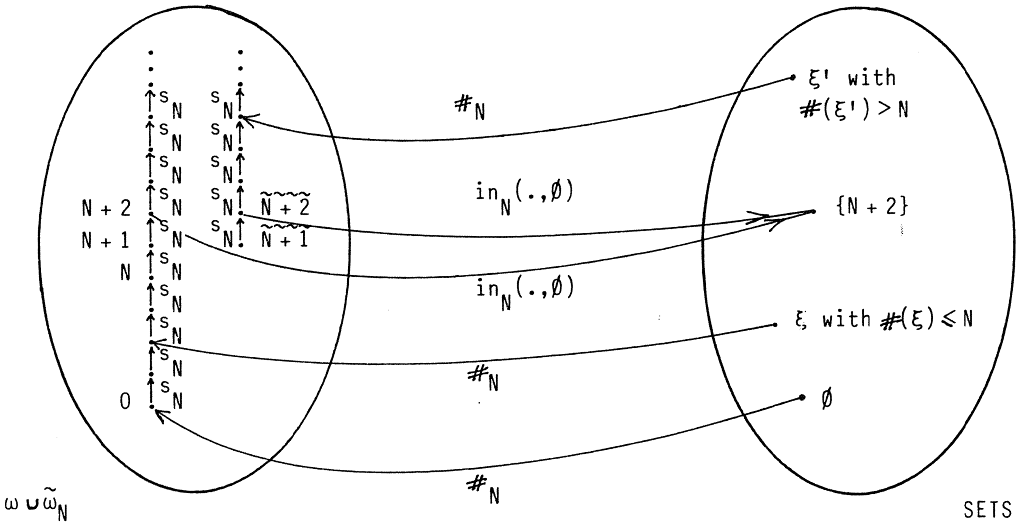
and IN names the function in_N: $((\omega \cup \tilde{\omega}_N) \times \text{SETS}) \to$ SETS, given by

$$in_N(x,\xi) = \xi \cup \{x\} \quad (x\in\omega, \xi\in\text{SETS})$$

and

$$in_N(\tilde{x},\xi) = \xi \cup \{x\} \quad (x\in\omega, x>N, \xi\in\text{SETS}).$$

$SOI^N_{\#}$ can be pictured as follows:



It is easy to see that $SOI_{\#}^{N} \models \equiv_N$, i.e. if two terms are $\equiv_N$ - congruent, then they are equal when evaluated in $SOI_{\#}^{N}$.

Take some second sort term τ_1 such that $val(\underline{\#}\tau_1) = N+1$. Then:

$$S^{N+1}(\underline{0}) \equiv_{N+1} \underline{\#}(\tau) \text{ (by definition of } \equiv_{N+1}\text{), but since}$$

$$SOI_{\#}^{N} \not\models S^{N+1}(\underline{0}) = \underline{\#}(\tau) \text{ we have that } S^{N+1}(\underline{0}) \not\equiv_N \underline{\#}(\tau).$$

Hence, $\equiv_N \neq \equiv_{N+1}$, and we have a strict inclusion. This completes the proof of Claim 1. Next we proceed to prove a second claim.

<u>Claim 2</u>: Suppose the length of an equation e is $\leq N_e$ for some $N_e \in \omega$. Then $SOI_{\#} \models e \Rightarrow \equiv_e \subseteq \equiv_{N_e}$, where $\equiv_e$ is the congruence relation induced by e.

Using claim 2 we can finish our proof as follows:

Let $N_0 = \max_{e \in E}$ length (e). Then for each $e \in E$, length (e) $\leq N_0$ and $SOI_{\#} \models e$, so $\equiv_e \subseteq \equiv_{N_0}$ for every $e \in E$. So $\equiv_E \subseteq \equiv_{N_0}$. However we know $\equiv_{N_0} \subsetneq \equiv_{SOI}$, so $\equiv_E \subsetneq \equiv_{SOI}$. Therefore, $T(\Sigma,E) = T(\Sigma)/\equiv_E \not\cong T(\Sigma)/\equiv_{SOI} \cong SOI_{\#}$, i.e. E can *not* specify $SOI_{\#}$.

The only thing left for us to do is to prove the last claim.

<u>Proof of Claim 2</u>: First we make a classification of the equations that may occur (taking into consideration that e is satisfied by $SOI_{\#}$). Equations of the kind $\tau_1(\vec{X},\vec{\Xi}) = \tau_1(\vec{X},\vec{\Xi})$ where $\tau_1(\vec{X},\vec{\Xi})$ and $\tau_2(\vec{X},\vec{\Xi})$ are second sort

polynomials are no problem: if we have such an equation e, and $(\tau,\tau') \in \equiv_e$, then $SOI_{\#} \models e$ implies that $val(\tau) = val(\tau')$, so $\tau \equiv_{SOI} \tau'$ and therefore we have that for all $N \in \omega$: $\tau \equiv_N \tau'$, and also $\tau \equiv_{N_e} \tau'$ for $N_e \geq$ length (e). Equations of the kind $t_1(\vec{X},\vec{\Xi}) = t_2(\vec{X},\vec{\Xi})$ (between first sort polynomials) are more problematical. In general, we can have them in the following forms

(a) $S^k(X_1) = S^\ell(X_2)$ where $k,\ell \in \omega$. (Including the special cases of $k = \ell$ and/or $X_1 = X_2$.)
(b) $S^k(\underline{0}) = S^\ell(\underline{0})$ with $k,\ell \in \omega$, possibly $k = \ell$.
(c) $S^k(X) = S^\ell(\underline{0})$ with $k,\ell \in \omega$.
(d) $S^k(X_1) = S^\ell(\underline{\#}(\tau(\vec{X},\vec{\Xi})))$ with $k,\ell \in \omega$ and $\tau(\vec{X},\vec{\Xi})$ a second sort polynomial.
(e) $S^k(\underline{0}) = S^\ell(\underline{\#}(\tau(\vec{X},\vec{\Xi})))$ with $k,\ell \in \omega$ and $\tau(\vec{X},\vec{\Xi})$ a second sort polynomial.
(f) $S^k(\underline{\#}(\tau_1(\vec{X},\vec{\Xi}))) = S^\ell(\underline{\#}(\tau_2(\vec{X},\vec{\Xi})))$ with $k,\ell \in \omega$ ($k = \ell$ possible!) and the $\tau_i(\vec{X},\vec{\Xi})$ second sort polynomials.

However, the requirement for equations to be satisfied by $SOI_{\#}$ leaves from the forms (a) to (c) only the following possibilities:

(α) $S^k(\underline{0}) = S^k(\underline{0})$ and $S^k(X) = S^k(X)$ with $k \in \omega$.

From (d) and (e) only the only remaining form is:

(β) $S^k(\underline{0}) = S^\ell(\underline{\#}(IN(t_1(\vec{X},\vec{\Xi}),\ldots,IN(t_n(\vec{X},\vec{\Xi}),\underline{\emptyset})\ldots)))$ with $k \geq \ell \geq 0$.

Next we ask ourselves how equations of the form (f) must be shaped in order to be satisfied by $SOI_{\#}$.

We name the second sort polynomial $\tau(\vec{\Xi})$ in the polynomial

$$\tau_i(\vec{X},\vec{\Xi}) = IN(t_1(\vec{X},\vec{\Xi}),\ldots,IN(t_n(\vec{X},\vec{\Xi}),\tau(\Xi))\ldots)$$

such that $\tau(\vec{\Xi})$ is $\underline{\emptyset}$ or a Ξ_j, the *root* of $\tau_i(\vec{X},\vec{\Xi})$. Now it is obvious that as far as (f) with $k \neq \ell$ is concerned the root on both sides of the equality sign may not differ, and we shall prove now that also the case in which both roots are the same Ξ_j must be ruled out. (<u>Claim 3</u>)

<u>Proof of Claim 3</u>: In this case (for some j)

$$\tau_1(\vec{X},\vec{\Xi}) = IN(t_1(\vec{X},\vec{\Xi}),\ldots,IN(t_n(\vec{X},\vec{\Xi}),\Xi_j))\ldots)$$

and

$$\tau_2(\vec{X},\vec{\Xi}) = IN(t_1'(\vec{X},\vec{\Xi}),\ldots,IN(t_m'(\vec{X},\vec{\Xi}),\Xi_j))\ldots).$$

Choose some terms $\vec{t}^0$ and $\tau_i^0(i\neq j)$ to substitute for the indeterminates $\vec{X}$ and $\Xi_i(i\neq j)$ respectively, i.e. substitute $\vec{t}_0 = (t_1^0,\ldots,t_p^0)$ for $\vec{X}$ and $\vec{\tau}_0(\Xi_j) = (\tau_1^0,\ldots,\tau_{j-1}^0, \Xi_j, \tau_{j+1}^0,\ldots,\tau_q^0)$ for $\vec{\Xi}$.
Note that Ξ_j must remain a free variable if and when it occurs.
For the $t_i(\vec{X},\vec{\Xi})$ and $t_i'(\vec{X},\vec{\Xi})$ respectively there are two possibilities: they contain Ξ_j as a root or they do not. In the former case it is easy to see that

$$\#val(\Xi_j) \le val(t_i(\vec{t}^0,\vec{\tau}^0(\Xi_j))) \le \#val(\Xi_j) + k_i$$

and

$$\#val(\Xi_j) \le val(t_i'(\vec{t}^0,\vec{\tau}^0(\Xi_j))) \le \#val(\Xi_j) + k_i'$$

for some fixed k_i, $k_i' \in \omega$, for each substitution τ_j^0 for the indeterminate Ξ_j.
In the latter case

$$0 \le val(t_i(\vec{t}^0,\vec{\tau}^0(\Xi_j))) \le \ell_i, \quad \text{and}$$
$$0 \le val(t_i'(\vec{t}^0,\vec{\tau}^0(\Xi_j))) \le \ell_i'$$

for some fixed ℓ_i, $\ell_i' \in \omega$, for each substitution τ_j^0 for the indeterminate Ξ_j.
Now take

$$m' = \max \{(k_i,k_j') \mid 1 \le i \le n,\ 1 \le j \le m\},$$

and

$$M = \max \{\ell_i,\ell_j') + 2 \mid 1 \le i \le n,\ 1 \le j \le m\}.$$

Furthermore let τ_j^0 be such that

$$val(\tau_j^0) = \{0,1,\ldots,M-2,M+m',M+m'+1,\ldots,M+2m'\}.$$

Thus

$$\#val(\tau_j^0) = M + m'.$$

Then (f) evaluated in $SOI_\#$ with $\vec{t}$ for $\vec{X}$ and $\vec{\tau}^0(\tau_j^0)$ for $\vec{\Xi}$ becomes:

$$k + \#val(\tau_j^0) = \ell + \#val(\tau_j^0),$$

i.e. $k + M + m' = \ell + M + m'$, which is obviously not true in $SOI_\#$ for $k \neq \ell$.
This completes the proof of claim 3.

Hence from (f) only the following possibilities remain:

(γ) $S^k(\underline{\#}(\tau_1(\vec{X},\vec{\Xi}))) = S^k(\underline{\#}(\tau_2(\vec{X},\vec{\Xi})))$ with $k \in \omega$, and

(δ) $S^k(\underline{\#}(IN(t_1(\vec{X},\vec{\Xi}),\ldots,IN(t_n(\vec{X},\vec{\Xi}),\underline{\emptyset})\ldots)$
$= S^{\ell}(\underline{\#}(IN(t'_1(\vec{X},\vec{\Xi}),\ldots,IN(t'_m(\vec{X},\vec{\Xi}),\underline{\emptyset})\ldots)))$ with $k,\ell \in \omega$, $k \neq \ell$.

So of the (first sort) equations only the possibilities (α), (β), (γ) and (δ) remain. To prove that $SOI_{\#} \models e$ implies $\equiv_e \subseteq \equiv_{N_e}$ for some $N_e \geq$ length (e) for these cases we proceed as follows:

(α) Suppose $(t,t') \in \equiv_E$ by means of an equation of form (α). Then trivially also $(t,t') \in \equiv_{N_e}$.

(β) Suppose $(t,t') \in \equiv_E$ by means of an equation e of form (β).
e: $S^{k_0}(\underline{0}) = S^{\ell_0}\underline{\#}IN(t_1(\vec{X},\vec{\Xi}),\ldots,IN(t_n(\vec{X},\vec{\Xi}),\underline{\emptyset})\ldots)$,
$t = S^{k_0}(\underline{0})$ and
$t' = S^{\ell_0}\underline{\#}IN(t_1(\vec{X}^0,\vec{\Xi}^0),\ldots,IN(t_n(\vec{X}^0,\vec{\Xi}^0),\underline{\emptyset})\ldots)$.
Then $SOI_{\#} \models e$ implies $val(t) = val(t')$, i.e.
$k_0 = \ell_0 + val(\underline{\#}IN(t_1(\vec{X}^0,\vec{\Xi}^0),\ldots,\underline{\emptyset})\ldots)$, so
$k_0 - \ell_0 = val(\underline{\#}IN(t_1(\vec{X}^0,\vec{\Xi}^0),\ldots,\underline{\emptyset})\ldots) \leq N_0$
which implies

$$S^{k_0-\ell_0}(\underline{0}) \equiv_{N_e} \underline{\#}IN(t_1(\vec{X}^0,\vec{\Xi}^0),\ldots,\underline{\emptyset})\ldots).$$

So also $S^{k_0}(\underline{0}) \equiv_{N_e} S^{\ell_0}\underline{\#}IN(t_1(\vec{X}^0,\vec{\Xi}^0),\ldots,\underline{\emptyset})\ldots)$ i.e. $t \equiv_{N_e} t'$.

(γ) Suppose $(t,t') \in \equiv_E$ by means of an equation e of form (γ):
e: $S^{k_0}\underline{\#}\tau_1(\vec{X},\vec{\Xi}) = S^{k_0}\underline{\#}\tau_2(\vec{X},\vec{\Xi})$,
$t = S^{k_0}\underline{\#}\tau_1$,
$t' = S^{k_0}\underline{\#}\tau_2$ (where τ_i are second sort terms).
$SOI_{\#} \models e$ implies $val(t) = val(t')$ i.e. $val(\underline{\#}\tau_1) + k_0 = val(\underline{\#}\tau_2) + k_0$.
So $val(\underline{\#}\tau_1) = val(\underline{\#}\tau_2)$.
Now we have two cases:
(1) $val(\underline{\#}\tau_i) > N_e (i=1,2)$.

(2) $\mathrm{val}(\underline{\#}\tau_i) \le N_e (i=1,2)$.

(1) implies directly: $\underline{\#}\tau_1 \equiv_{N_e} \underline{\#}\tau_2$ and (2) $\Rightarrow$ $\underline{\#}\tau_i \equiv_{SOI} S^k(\underline{0})$

where $k \le N_0$, so also $\underline{\#}\tau_1 \equiv_{N_e} \underline{\#}\tau_2$.

Thus always $\underline{\#}\tau_1 \equiv_{N_e} \underline{\#}\tau_2$, and as $\equiv_{N_e}$ is a congruence also:

$S^{k_0}\underline{\#}\tau_1 \equiv_{N_e} S^{k_0}\underline{\#}\tau_2$, i.e. $(t,t') \in \equiv_{N_e}$.

(δ) Suppose $(t,t') \in \equiv_E$ by means of an equation e of form (δ):

e: $S^{k_0}\underline{\#}\mathrm{IN}(t_1(\vec{X},\vec{\Xi}),\ldots,t_n(\vec{X},\vec{\Xi}),\underline{\emptyset}\ldots) = S^{\ell_0}\underline{\#}\mathrm{IN}(t_1'(\vec{X},\vec{\Xi}),\ldots,t_m'(\vec{X},\vec{\Xi}),\underline{\emptyset}\ldots)$.

$t = S^{k_0}\underline{\#}\mathrm{IN}(t_1(\vec{X}^0,\vec{\Xi}^0),\ldots,t_n(\vec{X}^0,\vec{\Xi}^0),\underline{\emptyset})\ldots)$

$t' = S^{\ell_0}\underline{\#}\mathrm{IN}(t_1'(\vec{X}^0,\vec{\Xi}^0),\ldots,t_m'(\vec{X}^0,\vec{\Xi}^0),\underline{\emptyset})\ldots)$.

Abbreviate $t = S^{k_0}\underline{\#}\tau_1$ and $t' = S^{\ell_0}\underline{\#}\tau_2$.

$SOI_\# \models e$ implies: $\mathrm{val}(t) = \mathrm{val}(t')$, i.e. $\#\mathrm{val}(\tau_1) + k_0 = \#\mathrm{val}(\tau_2) + \ell_0$.

As $\#\mathrm{val}(\tau_i) = n_1 \le N_e$ and $\#\mathrm{val}(\tau_2) = n_2 \le N_e$:

$\underline{\#}\tau_1 \equiv_{N_e} S^{n_1}(\underline{0})$ and $\underline{\#}\tau_2 \equiv_{N_e} S^{n_2}(\underline{0})$ with $n_1, n_2 \le N_e$.

So $S^{k_0}\underline{\#}\tau_1 \equiv_{N_e} S^{k_0+n_1}(\underline{0})$ and $S^{\ell_0}\underline{\#}\tau_2 \equiv_{N_e} S^{\ell_0+n_2}(\underline{0})$

So, as $k_0 + n_1 = \ell_0 + n_2$: $S^{k_0}\underline{\#}\tau_1 \equiv_{N_e} S^{k_0+n_1}(\underline{0}) \equiv_{N_e} S^{\ell_0+n_2}(\underline{0}) \equiv_{N_e} S^{\ell_0}\underline{\#}\tau_2$,

i.e. $t \equiv_{N_e} t'$. Therefore we always have that if t and t' are identified by some equation e that is satisfied by $SOI_\#$: $t \equiv_{N_e} t'$ for some $N_e \ge \mathrm{length}(e)$. This proves claim 2.

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